

FIXED POINTS SUBGROUPS $G^{\sigma, \sigma'}$ BY TWO INVOLUTIVE AUTOMORPHISMS σ, σ' OF EXCEPTIONAL COMPACT LIE GROUP G , PART II, $G = E_8$

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Dedicated to Professor Ichiro Yokota on the occasion of his eightieth birthday

ABSTRACT. For the simply connected compact exceptional Lie group E_8 , we determine the structure of subgroup $(E_8)^{\sigma, \sigma'}$ of E_8 which is the intersection $(E_8)^\sigma \cap (E_8)^{\sigma'}$. Then the space $E_8/(E_8)^{\sigma, \sigma'}$ is the exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric space of type EVIII-VIII-VIII, and that we give two involutions σ, σ' for the space $E_8/(E_8)^{\sigma, \sigma'}$ concretely.

0. Introduction

In the preceding paper [3], for the simply connected compact exceptional Lie groups $G = F_4, E_6$ and E_7 , we determined the structure of subgroups $G^{\sigma, \sigma'}$ of G which is the intersection $G^\sigma \cap G^{\sigma'}$. Their results were given as

$$\begin{aligned} (F_4)^{\sigma, \sigma'} &\cong Spin(8), \\ (E_6)^{\sigma, \sigma'} &\cong (U(1) \times U(1) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_4), \\ (E_7)^{\sigma, \sigma'} &\cong (SU(2) \times SU(2) \times SU(2) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2). \end{aligned}$$

In [1], a classification was given of the exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces G/K where G is an exceptional compact Lie groups and $Spin(8)$, and the structure of K was determined as Lie algebras. Our results correspond to the realizations of the exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces with the pair (σ, σ') of commuting involutions of F_4, E_6 and E_7 . To be specific, since the groups $(F_4)^{\sigma, \sigma'}$, $(E_6)^{\sigma, \sigma'}$ and $(E_7)^{\sigma, \sigma'}$ are connected from the results above, we see that the space $F_4/(F_4)^{\sigma, \sigma'}$, $E_6/(E_6)^{\sigma, \sigma'}$ and $E_7/(E_7)^{\sigma, \sigma'}$ are the exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces of type FII-II-II, EIII-III-III and EVI-VI-VI, respectively.

In the present paper, for the simply connected compact exceptional Lie group E_8 , we consider the subgroup $(E_8)^{\sigma, \sigma'}$ of E_8 . We showed in [4] that the Lie algebra $(\mathfrak{e}_8)^{\sigma, \sigma'}$ of $(E_8)^{\sigma, \sigma'}$ is isomorphic to $\mathfrak{so}(8) \oplus \mathfrak{so}(8)$ as a Lie algebra. So, it is main purpose to determine the structure of the subgroup $(E_8)^{\sigma, \sigma'}$. Our result is as follows:

$$(E_8)^{\sigma, \sigma'} \cong (Spin(8) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2).$$

The first $Spin(8)$ is $(E_8)^{\sigma, \sigma', \mathfrak{so}(8)} \subset (E_8)^{\sigma, \sigma'}$ and the second $Spin(8)$ is $(F_4)^{\sigma, \sigma'} \subset (E_8)^{\sigma, \sigma'}$, where the definition of the group $(E_8)^{\sigma, \sigma', \mathfrak{so}(8)}$ is showed in section 4. On the exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric spaces for E_8 , there exist four types ([1]). This amounts to the realization of one type of these four types with the pair (σ, σ') of commuting involutions of E_8 .

The essential part to prove this fact is to show the connectedness of the group $(E_8)^{\sigma, \sigma', \mathfrak{so}(8)}$. For this end, we need to treat the complex case, that is, we need the

2000 *Mathematics Subject Classification.* 22E10, 22E15, 53C35.
Key words and phrases. Exceptional Lie group, Symmetric space.

following facts.

$$\begin{aligned}
(F_4^C)^{\sigma, \sigma'} &\cong Spin(8, C), \\
(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8)} &\cong SL(2, C) \times SL(2, C) \times SL(2, C), \\
(E_7^C)^{\sigma, \sigma'} &\cong (SL(2, C) \times SL(2, C) \times SL(2, C) \times Spin(8, C)) / (\mathbf{Z}_2 \times \mathbf{Z}_2), \\
(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8)} &\cong Spin(8, C), \\
(E_8^C)^{\sigma, \sigma'} &\cong (Spin(8, C) \times Spin(8, C)) / (\mathbf{Z}_2 \times \mathbf{Z}_2)
\end{aligned}$$

and the connectedness of $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$. Even if some of their proofs are similar to the previous paper [5] and [6], we write in detail again.

To consider the group E_8 , we need to know some knowledge of the group E_7 . As for them we refer [6] and [7]. This paper is a continuation of [3]. We use the same notations as [3], [6] and [7].

1. Lie groups $F_4^C, F_4, E_6^C, E_7^C, E_8^C$ and E_8

We describe definitions of Lie groups used in this paper.

Let \mathfrak{J}^C and \mathfrak{J} be the exceptional C - and \mathbf{R} -Jordan algebras, respectively. The connected complex Lie group F_4^C and the connected compact Lie group F_4 are defined by

$$\begin{aligned}
F_4^C &= \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}, \\
F_4 &= \{\alpha \in \text{Iso}_R(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\},
\end{aligned}$$

respectively, and the simply connected complex Lie group E_6^C is given by

$$E_6^C = \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \det \alpha X = \det X\}.$$

We define \mathbf{R} -linear transformations σ and σ' of \mathfrak{J} by

$$\sigma X = \sigma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \quad \sigma' X = \begin{pmatrix} \xi_1 & x_3 & -\bar{x}_2 \\ \bar{x}_3 & \xi_2 & -x_1 \\ -x_2 & -\bar{x}_1 & \xi_3 \end{pmatrix},$$

respectively. Then $\sigma, \sigma' \in F_4 \subset F_4^C$. σ and σ' are commutative: $\sigma\sigma' = \sigma'\sigma$.

Let \mathfrak{P}^C be the Freudenthal C -vector space

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C,$$

in which the Freudenthal cross operation $P \times Q, P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, is defined as follow:

$$P \times Q = \Phi(\phi, A, B, \nu), \quad \begin{cases} \phi = -\frac{1}{2}(X \vee W + Z \vee Y) \\ A = -\frac{1}{4}(2Y \times W - \xi Z - \zeta X) \\ B = \frac{1}{4}(2X \times Z - \eta W - \omega Y) \\ \nu = \frac{1}{8}((X, W) + (Z, Y) - 3(\xi\omega + \zeta\eta)), \end{cases}$$

where $X \vee W \in \mathfrak{e}_6^C$ is defined by

$$X \vee W = [\tilde{X}, \tilde{W}] + (X \circ W - \frac{1}{3}(X, W)E)^\sim,$$

here $\tilde{X} : \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ is defined by $\tilde{X}Z = X \circ Z, Z \in \mathfrak{J}^C$.

Now, the simply connected complex Lie group E_7^C is defined by

$$E_7^C = \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}.$$

The Lie algebra \mathfrak{e}_7^C of the group E_7^C is given by

$$\mathfrak{e}_7^C = \{\Phi(\phi, A, B, \nu) \mid \phi \in \mathfrak{e}_6^C, A, B \in \mathfrak{J}^C, \nu \in C\}.$$

Naturally we have $F_4^C \subset E_6^C \subset E_7^C$. Finally, in a C -vector space \mathfrak{e}_8^C :

$$\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C,$$

we define a Lie bracket $[R_1, R_2]$ by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)] = (\Phi, P, Q, r, s, t),$$

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\ Q = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1 \\ P = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + t_1 P_2 - t_2 P_1 \\ r = -\frac{1}{8}\{P_1, Q_2\} + \frac{1}{8}\{P_2, Q_1\} + s_1 t_2 - s_2 t_1 \\ s = \frac{1}{4}\{P_1, P_2\} + 2r_1 s_2 - 2r_2 s_1 \\ t = -\frac{1}{4}\{Q_1, Q_2\} - 2r_1 t_2 + 2r_2 t_1, \end{cases}$$

where $\{P, Q\} = (X, W) - (Y, Z) + \xi\omega - \eta\zeta$, $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$, then \mathfrak{e}_8^C becomes a complex simple Lie algebra of type E_8 ([2]).

We define a C -linear transformation $\tilde{\lambda}$ of \mathfrak{e}_8^C by

$$\tilde{\lambda}(\Phi, P, Q, r, s, t) = (\lambda\Phi\lambda^{-1}, \lambda Q, -\lambda P, -r, -t, -s),$$

where λ is a C -linear transformation of \mathfrak{P}^C defined by $\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi)$.

The complex conjugation in \mathfrak{e}_8^C is denoted by τ :

$$\tau(\Phi, P, Q, r, s, t) = (\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau s, \tau t),$$

where τ in the right hand side is the usual complex conjugation in the complexification.

Now, the connected complex Lie group E_8^C and connected compact Lie group E_8 are given by

$$E_8^C = \{\alpha \in \text{Iso}_C(\mathfrak{e}_8^C) \mid \alpha[R, R'] = [\alpha R, \alpha R']\},$$

$$E_8 = \{\alpha \in E_8^C \mid \tau\tilde{\lambda}\alpha\tilde{\lambda}\tau = \alpha\} = (E_8^C)^{\tau\tilde{\lambda}},$$

respectively.

For $\alpha \in E_7^C$, the mapping $\tilde{\alpha} : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ is defined by

$$\tilde{\alpha}(\Phi, P, Q, r, s, t) = (\alpha\Phi\alpha^{-1}, \alpha P, \alpha Q, r, s, t),$$

then $\tilde{\alpha} \in E_8^C$, so α and $\tilde{\alpha}$ will be identified. The group E_8^C contains E_7^C as a subgroup by

$$\begin{aligned} E_7^C &= \{\tilde{\alpha} \in E_8^C \mid \alpha \in E_7^C\} \\ &= (E_8^C)_{(0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)}. \end{aligned}$$

Similarly, $E_7 \subset E_8$. In particular, elements σ, σ' of F_4 are also elements of $E_8 \subset E_8^C$. Therefore the actions of σ and σ' on \mathfrak{e}_8^C are given as

$$\begin{aligned} \sigma(\Phi, P, Q, r, s, t) &= (\sigma\Phi\sigma^{-1}, \sigma P, \sigma Q, r, s, t), \\ \sigma'(\Phi, P, Q, r, s, t) &= (\sigma'\Phi\sigma'^{-1}, \sigma' P, \sigma' Q, r, s, t). \end{aligned}$$

Hereafter, in \mathfrak{e}_8^C , we shall use the following notations.

$$\begin{aligned}\Phi &= (\Phi, 0, 0, 0, 0, 0), & P^- &= (0, P, 0, 0, 0, 0), & Q_- &= (0, 0, Q, 0, 0, 0), \\ \tilde{r} &= (0, 0, 0, r, 0, 0), & s^- &= (0, 0, 0, 0, s, 0), & t_- &= (0, 0, 0, 0, 0, t).\end{aligned}$$

2. The group $(F_4^C)^{\sigma, \sigma'}$

We shall investigate the structure of a subgroup $(F_4^C)^{\sigma, \sigma'}$ of the group F_4^C :

$$(F_4^C)^{\sigma, \sigma'} = \{\alpha \in F_4^C \mid \sigma\alpha = \alpha\sigma, \sigma'\alpha = \alpha\sigma'\}.$$

Lemma 2.1. *In the Lie algebra \mathfrak{f}_4^C of the group F_4^C :*

$$\mathfrak{f}_4^C = \{\delta \in \text{Hom}_C(\mathfrak{J}^C) \mid \delta(X \circ Y) = \delta X \circ Y + X \circ \delta Y\},$$

the Lie algebra $(\mathfrak{f}_4^C)^{\sigma, \sigma'}$ of the group $(F_4^C)^{\sigma, \sigma'}$ is isomorphic to $\mathfrak{spin}(8, C)$:

$$(\mathfrak{f}_4^C)^{\sigma, \sigma'} \cong \mathfrak{spin}(8, C) = \mathfrak{so}(8, C).$$

The action of $D = (D_1, D_2, D_3) \in \mathfrak{spin}(8, C)$ on \mathfrak{J}^C is given by

$$D \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3(x_3) & \overline{D_2(x_2)} \\ \overline{D_3(x_3)} & 0 & D_1(x_1) \\ D_2(x_2) & \overline{D_1(x_1)} & 0 \end{pmatrix},$$

where D_1, D_2, D_3 are elements of $\mathfrak{so}(8, C)$ satisfying

$$D_1(x)y + xD_2(y) = \overline{D_3(\overline{xy})}, \quad x, y \in \mathfrak{C}^C.$$

Theorem 2.2. $(F_4^C)^{\sigma, \sigma'} \cong Spin(8, C).$

Proof. Let $Spin(8, C)$ be the group defined by

$$\{(\alpha_1, \alpha_2, \alpha_3) \in SO(8, C) \times SO(8, C) \times SO(8, C) \mid (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3(\overline{xy})}, x, y \in \mathfrak{C}^C\}$$

(cf.[7] Theorem 1.47). Now, we define a mapping $\varphi : Spin(8, C) \rightarrow (F_4^C)^{\sigma, \sigma'}$ by

$$\varphi(\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix}.$$

φ is well-defined: $\varphi(\alpha_1, \alpha_2, \alpha_3) \in (F_4^C)^{\sigma, \sigma'}$. Clearly φ is a homomorphism and injective. Since $(F_4^C)^{\sigma, \sigma'} = ((F_4^C)^\sigma)^{\sigma'} = (Spin(9, C))^{\sigma'}$ ([6] Theorem 2.4.3) is connected and $\dim_C((\mathfrak{f}_4^C)^{\sigma, \sigma'}) = \dim_C(\mathfrak{spin}(8, C))$ (Lemma 2.1), φ is onto. Thus we have the required isomorphism $(F_4^C)^{\sigma, \sigma'} \cong Spin(8, C)$. \square

3. The groups $(E_7^C)^{\sigma, \sigma'}$ and $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$

The aim of this section is to show the connectedness of the group $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$. Now, we define subgroups $(E_7^C)^{\sigma, \sigma'}$ and $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ of the group E_7^C respectively by

$$\begin{aligned}(E_7^C)^{\sigma, \sigma'} &= \{\alpha \in E_7^C \mid \sigma\alpha = \alpha\sigma, \sigma'\alpha = \alpha\sigma'\}, \\ (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} &= \{\alpha \in (E_7^C)^{\sigma, \sigma'} \mid \Phi_D \alpha = \alpha \Phi_D \text{ for all } D \in \mathfrak{so}(8, C)\},\end{aligned}$$

where $\Phi_D = (D, 0, 0, 0) \in \mathfrak{e}_7^C$, $D \in \mathfrak{so}(8, C) = (\mathfrak{f}_4^C)^{\sigma, \sigma'}$.

Lemma 3.1. (1) The Lie algebra $(\mathfrak{e}_7^C)^{\sigma, \sigma'}$ of the group $(E_7^C)^{\sigma, \sigma'}$ is given by

$$\begin{aligned} & (\mathfrak{e}_7^C)^{\sigma, \sigma'} \\ &= \{ \Phi \in \mathfrak{e}_7^C \mid \sigma\Phi = \Phi\sigma, \sigma'\Phi = \Phi\sigma' \} \\ &= \left\{ \Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^C \mid \begin{array}{l} \phi \in (\mathfrak{e}_6^C)^{\sigma, \sigma'}, A, B \in \mathfrak{J}^C, \\ A, B \text{ are diagonal forms, } \nu \in C \end{array} \right\}. \end{aligned}$$

(2) The Lie algebra $(\mathfrak{e}_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ of the group $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is given by

$$\begin{aligned} & (\mathfrak{e}_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \\ &= \{ \Phi \in (\mathfrak{e}_7^C)^{\sigma, \sigma'} \mid [\Phi, \Phi_D] = 0 \text{ for all } D \in \mathfrak{so}(8, C) \} \\ &= \left\{ \Phi(\phi, A, B, \nu) \in (\mathfrak{e}_7^C)^{\sigma, \sigma'} \mid \begin{array}{l} \phi = \tilde{T}, T = \text{diag}(\tau_1, \tau_2, \tau_3) \in \mathfrak{J}^C, \tau_k \in C, \\ \tau_1 + \tau_2 + \tau_3 = 0, A = \text{diag}(\alpha_1, \alpha_2, \alpha_3) \in \mathfrak{J}^C, \\ \alpha_k \in C, B = \text{diag}(\beta_1, \beta_2, \beta_3) \in \mathfrak{J}^C, \beta_k \in C \end{array} \right\}. \end{aligned}$$

In particular, we have

$$\dim_C((\mathfrak{e}_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}) = 2 + 3 \times 2 + 1 = 9.$$

In the Lie algebra \mathfrak{e}_7^C , we define

$$\kappa = \Phi(-2E_1 \vee E_1, 0, 0, -1), \quad \mu = \Phi(0, E_1, E_1, 0),$$

and we define the group $(E_7^C)^{\kappa, \mu}$ by

$$(E_7^C)^{\kappa, \mu} = \{ \alpha \in E_7^C \mid \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu \}.$$

Note that if $\alpha \in E_7^C$ satisfies $\kappa\alpha = \alpha\kappa$, then α automatically satisfies $\sigma\alpha = \alpha\sigma$ because $-\sigma = \exp(\pi i \kappa)$, $i \in C$, $i^2 = -1$.

Lemma 3.2. For $A \in SL(2, C) = \{A \in M(2, C) \mid \det A = 1\}$, we define C -linear transformations $\phi_k(A)$, $k = 1, 2, 3$ of \mathfrak{P}^C by

$$\begin{aligned} & \phi_k(A) \left(\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, \xi, \eta \right) \\ &= \left(\begin{pmatrix} \xi'_1 & x'_3 & \bar{x}'_2 \\ \bar{x}'_3 & \xi'_2 & x'_1 \\ x'_2 & \bar{x}'_1 & \xi'_3 \end{pmatrix}, \begin{pmatrix} \eta'_1 & y'_3 & \bar{y}'_2 \\ \bar{y}'_3 & \eta'_2 & y'_1 \\ y'_2 & \bar{y}'_1 & \eta'_3 \end{pmatrix}, \xi', \eta' \right), \\ & \begin{pmatrix} \xi'_k \\ \eta'_k \end{pmatrix} = A \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \begin{pmatrix} \xi'_{k+1} \\ \eta'_{k+1} \end{pmatrix} = A \begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix}, \begin{pmatrix} \xi'_{k+2} \\ \eta'_{k+2} \end{pmatrix} = A \begin{pmatrix} \xi_{k+2} \\ \eta_{k+2} \end{pmatrix}, \\ & \begin{pmatrix} x'_k \\ y'_k \end{pmatrix} = {}^t A^{-1} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \begin{pmatrix} x'_{k+1} \\ y'_{k+1} \end{pmatrix} = \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix}, \begin{pmatrix} x'_{k+2} \\ y'_{k+2} \end{pmatrix} = \begin{pmatrix} x_{k+2} \\ y_{k+2} \end{pmatrix} \end{aligned}$$

(where indices are considered as mod 3). Then $\phi_k(A) \in (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \subset (E_7^C)^{\sigma, \sigma'}$. Moreover, $\phi_k(A) \in ((E_7^C)^{\kappa, \mu})^{\sigma', \mathfrak{so}(8, C)} \subset ((E_7^C)^{\kappa, \mu})^{\sigma'}$ for $k = 2, 3$.

Proposition 3.3. $(E_7^C)^\sigma \cong (SL(2, C) \times Spin(12, C))/\mathbf{Z}_2$, $\mathbf{Z}_2 = \{(E, 1), (-E, -\sigma)\}$.

Proof. Let $Spin(12, C) = (E_7^C)^{\kappa, \mu}$ ([6] Proposition 4.6.10) $\subset (E_7^C)^\sigma$. We define a mapping $\varphi_1 : SL(2, C) \times Spin(12, C) \rightarrow (E_7^C)^\sigma$ by

$$\varphi_1(A_1, \delta) = \phi_1(A_1)\delta.$$

Then we have this proposition (see [6] Theorem 4.6.13 for details). \square

Lemma 3.4. *The Lie algebra $((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma'}$ of the group $((E_7^C)^{\kappa,\mu})^{\sigma'}$ is given by*

$$\begin{aligned} & ((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma'} \\ &= \{ \Phi \in \mathfrak{e}_7^C \mid \kappa\Phi = \Phi\kappa, \mu\Phi = \Phi\mu, \sigma'\Phi = \Phi\sigma' \} \\ &= \left\{ \Phi(\phi, A, B, \mu) \in \mathfrak{e}_7^C \left| \begin{array}{l} \phi \in (\mathfrak{e}_6^C)^{\sigma,\sigma'}, A, B \in \mathfrak{J}^C, \sigma A = \sigma' A = A, \\ (E_1, A) = 0, \sigma B = \sigma' B = B, (E_1, B) = 0, \\ \nu = -\frac{3}{2}(\phi E_1, E_1) \end{array} \right. \right\}. \end{aligned}$$

In particular, we have

$$\dim_C(((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma'}) = 30 + 2 \times 2 = 34.$$

Proposition 3.5. $((E_7^C)^{\kappa,\mu})^{\sigma'} \cong (SL(2, C) \times SL(2, C) \times Spin(8, C))/\mathbf{Z}_2, \mathbf{Z}_2 = \{(E, E, 1), (-E, -E, \sigma)\}$.

Proof. Let $Spin(8, C) = (F_4^C)^{\sigma,\sigma'} = (Spin(9, C))^{\sigma'} \subset (Spin(12, C))^{\sigma'} = ((E_7^C)^{\kappa,\mu})^{\sigma'}$. Now, we define a mapping $\varphi_2 : SL(2, C) \times SL(2, C) \times Spin(8, C) \rightarrow ((E_7^C)^{\kappa,\mu})^{\sigma'}$ by

$$\varphi_2(A_2, A_3, \beta) = \phi_2(A_2)\phi_3(A_3)\beta.$$

φ_2 is well-defined: $\varphi_2(A_2, A_3, \beta) \in ((E_7^C)^{\kappa,\mu})^{\sigma'}$ (Lemma 3.2). Since $\phi_2(A_2), \phi_3(A_3)$ and β commute with each other, φ_2 is a homomorphism. $\text{Ker } \varphi_2 = \{(E, E, 1), (-E, -E, \sigma)\} = \mathbf{Z}_2$. Since $((E_7^C)^{\kappa,\mu})^{\sigma'} = (Spin(12, C))^{\sigma'}$ is connected and $\dim_C(((\mathfrak{e}_7^C)^{\kappa,\mu})^{\sigma'}) = 34$ (Lemma 3.4) $= 3 + 3 + 28 = \dim_C(\mathfrak{sl}(2, C) \oplus \mathfrak{sl}(2, C) \oplus \mathfrak{so}(8, C))$, φ_2 is onto. Thus we have the required isomorphism $((E_7^C)^{\kappa,\mu})^{\sigma'} \cong (SL(2, C) \times SL(2, C) \times Spin(8, C))/\mathbf{Z}_2$. \square

Theorem 3.6. $(E_7^C)^{\sigma,\sigma'} \cong (SL(2, C) \times SL(2, C) \times SL(2, C) \times Spin(8, C))/(\mathbf{Z}_2 \times \mathbf{Z}_2), \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(E, E, E, 1), (E, -E, -E, \sigma)\} \times \{(E, E, E, 1), (-E, -E, E, \sigma')\}$.

Proof. Let $Spin(8, C) = (F_4^C)^{\sigma,\sigma'} = (E_7^C)^{\sigma,\sigma'}$ (Theorem 2.2) $\subset (E_7^C)^{\sigma,\sigma'}$. We define a mapping $\varphi : SL(2, C) \times SL(2, C) \times SL(2, C) \times Spin(8, C) \rightarrow (E_7^C)^{\sigma,\sigma'}$ by

$$\varphi(A_1, A_2, A_3, \beta) = \phi_1(A_1)\phi_2(A_2)\phi_3(A_3)\beta.$$

φ is well-defined: $\varphi(A_1, A_2, A_3, \beta) \in (E_7^C)^{\sigma,\sigma'}$ (Lemma 3.2). Since $\phi_1(A_1), \phi_2(A_2), \phi_3(A_3)$ and β commute with each other, φ is a homomorphism. We shall show that φ is onto. For $\alpha \in (E_7^C)^{\sigma,\sigma'} \subset (E_7^C)^{\sigma}$, there exist $A_1 \in SL(2, C)$ and $\delta \in Spin(12, C)$ such that $\alpha = \phi_1(A_1)\delta$ (Proposition 3.3). From the condition $\sigma'\alpha\sigma' = \alpha$, that is, $\sigma'\varphi_1(A_1, \delta)\sigma' = \varphi_1(A_1, \delta)$, we have $\varphi_1(A_1, \sigma'\delta\sigma') = \varphi_1(A_1, \delta)$. Hence

$$\left\{ \begin{array}{l} A_1 = A_1 \\ \sigma'\delta\sigma' = \delta, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} A_1 = -A_1 \\ \sigma'\delta\sigma' = -\sigma\delta. \end{array} \right.$$

The latter case is impossible because $A_1 = -A_1$ is false. In the first case, from $\sigma'\delta\sigma' = \delta$, we have $\delta \in (Spin(12, C))^{\sigma'} = ((E_7^C)^{\kappa,\mu})^{\sigma'}$. Hence there exist $A_2, A_3 \in SL(2, C)$ and $\beta \in Spin(8, C)$ such that $\delta = \phi_2(A_2)\phi_3(A_3)\beta$ (Proposition 3.5). Then

$$\alpha = \phi_1(A_1)\delta = \phi_1(A_1)\phi_2(A_2)\phi_3(A_3)\beta = \varphi(A_1, A_2, A_3, \beta).$$

Therefore φ is onto. It is not difficult to see that

$$\begin{aligned} \text{Ker } \varphi &= \{(E, E, E, 1), (E, -E, -E, \sigma), (-E, E, -E, \sigma\sigma'), (-E, -E, E, \sigma')\} \\ &= \{(E, E, E, 1), (E, -E, -E, \sigma)\} \times \{(E, E, E, 1), (-E, -E, E, \sigma')\} \\ &= \mathbf{Z}_2 \times \mathbf{Z}_2. \end{aligned}$$

Thus we have the required isomorphism $(E_7^C)^{\sigma, \sigma'} \cong (SL(2, C) \times SL(2, C) \times SL(2, C) \times Spin(8, C)) / (Z_2 \times Z_2)$. \square

Now, we determine the structure of the subgroup $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ of E_7^C .

Theorem 3.7. $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \cong SL(2, C) \times SL(2, C) \times SL(2, C)$.

In particular, $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is connected.

Proof. We define a mapping $\varphi : SL(2, C) \times SL(2, C) \times SL(2, C) \rightarrow (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ by

$$\varphi(A_1, A_2, A_3) = \phi_1(A_1)\phi_2(A_2)\phi_3(A_3).$$

(φ is the restricted mapping of φ of Theorem 3.6). φ is well-defined: $\varphi(A_1, A_2, A_3) \in (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ (Lemma 3.2). Since $\dim_C(\mathfrak{sl}(2, C) \oplus \mathfrak{sl}(2, C) \oplus \mathfrak{sl}(2, C)) = 3 + 3 + 3 = 9 = \dim_C((\mathfrak{e}_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})$ (Lemma 3.1), $\text{Ker } \varphi$ is discrete. Hence $\text{Ker } \varphi$ is contained in the center $z(SL(2, C) \times SL(2, C) \times SL(2, C)) = \{(\pm E, \pm E, \pm E)\}$. However, φ maps these elements to $\pm 1, \pm \sigma, \pm \sigma', \pm \sigma\sigma'$. Hence $\text{Ker } \varphi = \{(E, E, E)\}$, that is, φ is injective. Finally, we shall show that φ is onto. For $\alpha \in (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \subset (E_7^C)^{\sigma, \sigma'}$, there exist $A_1, A_2, A_3 \in SL(2, C)$ and $\beta \in Spin(8, C)$ such that $\alpha = \varphi(A_1, A_2, A_3, \beta)$ (Theorem 3.6). From $\Phi_D \alpha = \alpha \Phi_D$, we have $\Phi_D \beta = \beta \Phi_D$ for all $D \in \mathfrak{so}(8, C)$, that is, β is contained in the center $z((F_4^C)^{\sigma, \sigma'}) = z(Spin(8, C)) = \{1, \sigma, \sigma', \sigma\sigma'\}$. However, $\sigma = \phi_1(E)\phi_2(-E)\phi_3(-E)$, $\sigma' = \phi_1(-E)\phi_2(-E)\phi_3(E)$, so $1, \sigma, \sigma', \sigma\sigma' \in \phi_1(SL(2, C))\phi_2(SL(2, C))\phi_3(SL(2, C))$. Hence we see that φ is onto. Thus we have the required isomorphism $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \cong SL(2, C) \times SL(2, C) \times SL(2, C)$. \square

4. Connectedness of the group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$

We define a subgroup $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ of the group $(E_8^C)^{\sigma, \sigma'}$ by

$$(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} = \left\{ \alpha \in E_8^C \mid \begin{array}{l} \sigma\alpha = \alpha\sigma, \sigma'\alpha = \alpha\sigma', \\ \Theta(R_D)\alpha = \alpha\Theta(R_D) \text{ for all } D \in \mathfrak{so}(8, C) \end{array} \right\},$$

where $R_D = (\Phi_D, 0, 0, 0, 0, 0) \in \mathfrak{e}_8^C$ and $\Theta(R_D)$ means $\text{ad}(R_D)$. Hereafter for $R \in \mathfrak{e}_8^C$, we denote $\text{ad}(R)$ by $\Theta(R)$.

To prove the connectedness of the group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$, we use the method used in [5]. However, we write this method in detail again. Firstly, we consider a subgroup $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$ of $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$:

$$((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-} = \{\alpha \in (E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \mid \alpha 1_- = 1_-\}.$$

Lemma 4.1. (1) The Lie algebra $((\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$ of the group $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$ is given by

$$\begin{aligned} & ((\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-} \\ &= \left\{ R \in \mathfrak{e}_8^C \mid \begin{array}{l} \sigma R = R, \sigma' R = R, \\ [R_D, R] = 0 \text{ for all } D \in \mathfrak{so}(8, C), [R, 1_-] = 0 \end{array} \right\} \\ &= \left\{ (\Phi, 0, Q, 0, 0, t) \in \mathfrak{e}_8^C \mid \begin{array}{l} \Phi \in (\mathfrak{e}_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}, Q = (Z, W, \zeta, \omega), \\ Z, W \text{ are diagonal forms, } \zeta, \omega, t \in C \end{array} \right\}. \end{aligned}$$

In particular,

$$\dim_C(((\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}) = 9 + 8 + 1 = 18.$$

(2) The Lie algebra $(\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ of the group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is given by

$$\begin{aligned} & (\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \\ &= \left\{ R \in \mathfrak{e}_8^C \mid \begin{array}{l} \sigma R = R, \sigma' R = R, \\ [R_D, R] = 0 \text{ for all } D \in \mathfrak{so}(8, C) \end{array} \right\} \\ &= \left\{ (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C \mid \begin{array}{l} \Phi \in (\mathfrak{e}_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}, \\ P = (X, Y, \xi, \eta), Q = (Z, W, \zeta, \omega), \\ X, Y, Z, W \text{ are diagonal forms,} \\ \xi, \eta, \zeta, \omega, r, u, v \in C \end{array} \right\}. \end{aligned}$$

In the following proposition, we denote by $(\mathfrak{P}^C)_d$ the subspace of \mathfrak{P}^C :

$$(\mathfrak{P}^C)_d = \{(X, Y, \xi, \eta) \in \mathfrak{P}^C \mid X, Y \text{ are diagonal forms, } \xi, \eta \in C\}.$$

Proposition 4.2. *The group $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$ is a semi-direct product of groups $\exp(\Theta((\mathfrak{P}^C)_d \oplus C_-))$ and $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$:*

$$((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-} = \exp(\Theta((\mathfrak{P}^C)_d \oplus C_-)) \cdot (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}.$$

In particular, $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$ is connected.

Proof. Let $((\mathfrak{P}^C)_d)_- \oplus C_- = \{(0, 0, Q, 0, 0, t) \mid Q \in (\mathfrak{P}^C)_d, t \in C\}$ be the subalgebra of $((\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$ (Lemma 4.1.(1)). From $[Q_-, t_-] = 0$, $\Theta(Q_-)$ commutes with $\Theta(t_-)$. Hence we have $\exp(\Theta(Q_- + t_-)) = \exp(\Theta(Q_-)) \exp(\Theta(t_-))$. Therefore $\exp(\Theta((\mathfrak{P}^C)_d)_- \oplus C_-)$ is a subgroup of $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$. Now, let $\alpha \in ((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$ and set

$$\alpha \tilde{1} = (\Phi, P, Q, r, s, t), \quad \alpha 1^- = (\Phi_1, P_1, Q_1, r_1, s_1, t_1).$$

Then from the relation $[\alpha \tilde{1}, 1_-] = \alpha[\tilde{1}, 1_-] = -2\alpha 1_- = -21_-, [\alpha 1^-, 1_-] = \alpha[1^-, 1_-] = \alpha \tilde{1}$, we have

$$P = 0, s = 0, r = 1, \Phi = 0, P_1 = -Q, s_1 = 1, r_1 = -\frac{t}{2}.$$

Moreover, from $[\alpha \tilde{1}, \alpha 1^-] = \alpha[\tilde{1}, 1^-] = 2\alpha 1^-$, we have

$$\Phi_1 = \frac{1}{2}Q \times Q, Q_1 = -\frac{t}{2}Q - \frac{1}{3}\Phi_1 Q, t_1 = -\frac{t^2}{4} - \frac{1}{16}\{Q, Q_1\}.$$

So, α is of the form

$$\alpha = \begin{pmatrix} * & * & * & 0 & \frac{1}{2}Q \times Q & 0 \\ * & * & * & 0 & -Q & 0 \\ * & * & * & Q & -\frac{t}{2}Q - \frac{1}{6}(Q \times Q)Q & 0 \\ * & * & * & 1 & -\frac{t}{2} & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & t & -\frac{t^2}{4} + \frac{1}{96}\{Q, (Q \times Q)Q\} & 1 \end{pmatrix}.$$

On the other hand, we have

$$\begin{aligned} \delta 1^- &= \exp\left(\Theta\left(\left(\frac{t}{2}\right)_-\right)\right) \exp(\Theta(Q_-)) 1^- \\ &= \begin{pmatrix} \frac{1}{2}Q \times Q \\ -Q \\ -\frac{t}{2}Q - \frac{1}{6}(Q \times Q)Q \\ -\frac{t}{2} \\ 1 \\ -\frac{t^2}{4} + \frac{1}{96}\{Q, (Q \times Q)Q\} \end{pmatrix} = \alpha 1^-, \end{aligned}$$

and also we get

$$\delta \tilde{1} = \alpha \tilde{1}, \quad \delta 1_- = \alpha 1_-$$

easily. Hence we see that $\delta^{-1}\alpha \in ((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{\tilde{1}, 1^-, 1_-} = (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$. Therefore we have

$$((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1_-} = \exp(\Theta(((\mathfrak{P}^C)_d)_- \oplus C_-))(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}.$$

Furthermore, for $\beta \in (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$, it is easy to see that

$$\beta(\exp(\Theta(Q_-)))\beta^{-1} = \exp(\Theta(\beta Q_-)), \quad \beta((\exp(\Theta(t_-)))\beta^{-1} = \exp(\Theta(t_-)).$$

This shows that $\exp(\Theta(((\mathfrak{P}^C)_d)_- \oplus C_-)) = \exp(\Theta(((\mathfrak{P}^C)_d)_-))\exp(\Theta(C_-))$ is a normal subgroup of $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1_-}$. Moreover, we have a split exact sequence

$$1 \rightarrow \exp(\Theta(((\mathfrak{P}^C)_d)_- \oplus C_-)) \rightarrow ((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1_-} \rightarrow (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \rightarrow 1.$$

Hence $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1_-}$ is a semi-direct product of $\exp(\Theta(((\mathfrak{P}^C)_d)_- \oplus C_-))$ and $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$:

$$((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1_-} = \exp(\Theta(((\mathfrak{P}^C)_d)_- \oplus C_-)) \cdot (E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}.$$

$\exp(\Theta(((\mathfrak{P}^C)_d)_- \oplus C_-))$ is connected and $(E_7^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is connected (Theorem 3.7), hence $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1_-}$ is also connected. \square

For $R \in \mathfrak{e}_8^C$, we define a C -linear mapping $R \times R : \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ by

$$(R \times R)R_1 = [R, [R, R_1]] + \frac{1}{30}B_8(R, R_1)R, \quad R_1 \in \mathfrak{e}_8^C$$

(where B_8 is the Killing form of the Lie algebra \mathfrak{e}_8^C) and a space \mathfrak{W}^C by

$$\mathfrak{W}^C = \{R \in \mathfrak{e}_8^C \mid R \times R = 0, R \neq 0\}.$$

Moreover, we define a subspace $(\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$ of \mathfrak{W}^C by

$$(\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)} = \{R \in \mathfrak{W}^C \mid \sigma R = R, \sigma' R = R, [R_D, R] = 0 \text{ for all } D \in \mathfrak{so}(8, C)\}.$$

Lemma 4.3. *For $R = (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C$ satisfying $\sigma R = R, \sigma' R = R$ and $[R_D, R] = 0$ for all $D \in \mathfrak{so}(8, C), R \neq 0$, R belongs to $(\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$ if and only if R satisfies the following conditions.*

- (1) $2s\Phi - P \times P = 0$ (2) $2t\Phi + Q \times Q = 0$ (3) $2r\Phi + P \times Q = 0$
- (4) $\Phi P - 3rP - 3sQ = 0$ (5) $\Phi Q + 3rQ - 3tP = 0$
- (6) $\{P, Q\} - 16(st + r^2) = 0$
- (7) $2(\Phi P \times Q_1 + 2P \times \Phi Q_1 - rP \times Q_1 - sQ \times Q_1) - \{P, Q_1\}\Phi = 0$

$$\begin{aligned}
(8) \quad & 2(\Phi Q \times P_1 + 2Q \times \Phi P_1 + rQ \times P_1 - tP \times P_1) - \{Q, P_1\}\Phi = 0 \\
(9) \quad & 8((P \times Q_1)Q - stQ_1 - r^2Q_1 - \Phi^2Q_1 + 2r\Phi Q_1) + 5\{P, Q_1\}Q - 2\{Q, Q_1\}P = 0 \\
(10) \quad & 8((Q \times P_1)P + stP_1 + r^2P_1 + \Phi^2P_1 + 2r\Phi P_1) + 5\{Q, P_1\}P - 2\{P, Q_1\}Q = 0 \\
(11) \quad & 18(\text{ad } \Phi)^2\Phi_1 + Q \times \Phi_1P - P \times \Phi_1Q + B_7(\Phi, \Phi_1)\Phi = 0 \\
(12) \quad & 18(\Phi_1\Phi P - 2\Phi\Phi_1P - r\Phi_1P - s\Phi_1Q) + B_7(\Phi, \Phi_1)P = 0 \\
(13) \quad & 18(\Phi_1\Phi Q - 2\Phi\Phi_1Q + r\Phi_1Q - t\Phi_1P) + B_7(\Phi, \Phi_1)Q = 0,
\end{aligned}$$

(where B_7 is the Killing form of the Lie algebra \mathfrak{e}_7^C) for all $\Phi_1 \in \mathfrak{e}_7^C, P_1, Q_1 \in \mathfrak{P}^C$.

We denote the connected component of $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ containing the unit element by $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_0$.

Proposition 4.4. *The group $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_0$ acts on $(\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$ transitively.*

Proof. Since $\alpha \in (E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ leaves invariant the Killing form B_8 of \mathfrak{e}_8^C : $B_8(\alpha R_1, \alpha R_2) = B_8(R_1, R_2)$, $R_k \in \mathfrak{e}_8^C, k = 1, 2$, we have $\alpha R \in (\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$, for $R \in (\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$. In fact,

$$\begin{aligned}
(\alpha R \times \alpha R)R_1 &= [\alpha R, [\alpha R, \alpha R_1]] + (1/30)B_8(\alpha R, R_1)\alpha R \\
&= \alpha[[R, [R, \alpha^{-1}R_1]] + (1/30)B_8(R, \alpha^{-1}R_1)\alpha R \\
&= \alpha((R \times R)\alpha^{-1}R_1 \\
&= 0, \\
[R_D, \alpha R] &= \alpha[\alpha^{-1}R_D, R] = \alpha[R_D, R] = 0.
\end{aligned}$$

This shows that $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ acts on $(\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$. We will show that this action is transitive. Firstly for $R_1 \in \mathfrak{e}_8^C$, since

$$\begin{aligned}
(1_- \times 1_-)R_1 &= [1_-, [1_-, (\Phi_1, P_1, Q_1, r_1, s_1, t_1)]] + (1/30)B_8(1_-, R_1)1_- \\
&= [1_-, (0, 0, P_1, -s_1, 0, 2r_1)] + 2s_11_- \\
&= (0, 0, 0, 0, -2s_1) + 2s_11_- \\
&= 0, \\
[R_D, 1_-] &= 0,
\end{aligned}$$

we have $1_- \in (\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$. In order to prove the transitivity, it is sufficient to show that any element $R \in (\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$ can be transformed to $1_- \in (\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$ by a certain $\alpha \in (E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$.

Case (1). $R = (\Phi, P, Q, r, s, t), t \neq 0$. From (2),(5),(6) of Lemma 4.3, we have

$$\Phi = -\frac{1}{2t}Q \times Q, \quad P = \frac{r}{t}Q - \frac{1}{6t^2}(Q \times Q)Q, \quad s = -\frac{r^2}{t} + \frac{1}{96t^3}\{Q, (Q \times Q)Q\}.$$

Now, for $\Theta = \Theta(0, P_1, 0, r_1, s_1, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})$ (Lemma 4.1.(2)), we compute $\Theta^n 1_-$,

$$\Theta^n 1_- = \begin{pmatrix} ((-2)^{n-1} + (-1)^n) r_1^{n-2} P_1 \times P_1 \\ \left((-2)^{n-1} - \frac{1+(-1)^{n-1}}{2} \right) r_1^{n-2} s_1 P_1 + \left(\frac{1-(-2)^n}{6} + \frac{(-1)^n}{2} \right) r_1^{n-3} (P_1 \times P_1) P_1 \\ ((-2)^n + (-1)^{n-1}) r_1^{n-1} P_1 \\ (-2)^{n-1} r_1^{n-1} s_1 \\ -((-2)^{n-2} + 2^{n-2}) r_1^{n-2} s_1^2 + \frac{2^{n-2} + (-2)^{n-2} - (-1)^{n-1}}{24} r_1^{n-4} \{P_1, (P_1 \times P_1) P_1\} \\ (-2)^n r_1^n \end{pmatrix}.$$

Hence by simple computing, we have

$$\exp(\Phi(0, P_1, 0, r_1, s_1, 0)) 1_- = (\exp \Theta) 1_- = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \Theta^n \right) 1_- = \begin{pmatrix} -\frac{1}{2r_1^2} (e^{-2r_1} - 2e^{-r_1} + 1) P_1 \times P_1 \\ \frac{s_1}{2r_1^2} (-e^{-2r_1} - e^{r_1} + e^{-r_1} + 1) P_1 + \frac{1}{6r_1^3} (-e^{-2r_1} + e^{r_1} + 3e^{-r_1} - 3) (P_1 \times P_1) P_1 \\ \frac{1}{r_1} (e^{-2r_1} - e^{-r_1}) P_1 \\ \frac{s_1}{2r_1} (1 - e^{-2r_1}) \\ -\frac{s_1^2}{4r_1^2} (e^{-2r_1} + e^{2r_1} - 2) + \frac{1}{96r_1^4} (e^{2r_1} + e^{-2r_1} - 4e^{r_1} - 4e^{-r_1} + 6) \{P_1, (P_1 \times P_1) P_1\} \\ e^{-2r_1} \end{pmatrix}.$$

(if $r_1 = 0, \frac{f(r_1)}{r_1^k}$ means $\lim_{r_1 \rightarrow 0} \frac{f(r_1)}{r_1^k}$). Here we set

$$Q = \frac{1}{r_1} (e^{-2r_1} - e^{-r_1}) P_1, \quad r = \frac{s_1}{2r_1} (1 - e^{-2r_1}), \quad t = e^{-2r_1}.$$

Then we have

$$(\exp \Theta) 1_- = \begin{pmatrix} -\frac{1}{2t} Q \times Q \\ \frac{r}{t} Q - \frac{1}{6t^2} (Q \times Q) Q \\ Q \\ r \\ -\frac{r^2}{t} + \frac{1}{96t^3} \{Q, (Q \times Q) Q\} \\ t \end{pmatrix} = \begin{pmatrix} \Phi \\ P \\ Q \\ r \\ s \\ t \end{pmatrix} = R.$$

Thus R is transformed to 1_- by $(\exp \Theta)^{-1} \in ((E_8^C)^{\sigma, \sigma', \text{so}(8, C)})_0$.

Case (2). $R = (\Phi, P, Q, r, s, 0), s \neq 0$. For $\lambda' = \exp(\Theta(0, 0, 0, 0, \frac{\pi}{2}, -\frac{\pi}{2})) \in ((E_8^C)^{\sigma, \sigma', \text{so}(8, C)})_0$ (Lemma 4.1.(2)), we have

$$\lambda' R = \lambda' (\Phi, P, Q, r, s, 0) = (\Phi, Q, -P, -r, 0, -s), \quad -s \neq 0.$$

So, this case can be reduced to Case (1).

Case (3). $R = (\Phi, P, Q, r, 0, 0), r \neq 0$. From (2), (5), (6) of Lemma 4.3, we have

$$Q \times Q = 0, \quad \Phi Q = -3rQ, \quad \{P, Q\} = 16r^2.$$

Now, for $\Theta = \Theta(0, Q, 0, 0, 0, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma, \sigma', \text{so}(8, C)})$ (Lemma 4.1.(2)), we see

$$(\exp \Theta) R = (\Phi, P + 2rQ, Q, r, -4r^2, 0), \quad -4r^2 \neq 0.$$

So, this case can be reduced to Case (2).

Case (4). $R = (\Phi, P, Q, 0, 0, 0)$, $Q \neq 0$. Choose $P_1 \in (\mathfrak{P}^C)_d$ such that $\{P_1, Q\} \neq 0$. For $\Theta = \Theta(0, P_1, 0, 0, 0, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})$ (Lemma 4.1.(2)), we have

$$(\exp \Theta)R = \left(*, *, *, -\frac{1}{8}\{P_1, Q\}, *, * \right).$$

So, this case can be reduced to the Case (3).

Case (5). $R = (\Phi, P, 0, 0, 0, 0)$, $P \neq 0$. Choose $Q_1 \in (\mathfrak{P}^C)_d$ such that $\{P, Q_1\} \neq 0$. For $\Theta = \Theta(0, 0, Q_1, 0, 0, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})$ (Lemma 4.1.(2)), we have

$$(\exp \Theta)R = \left(*, *, *, \frac{1}{8}\{P, Q_1\}, *, * \right).$$

So, this case can be reduced to the Case (3).

Case (6). $R = (\Phi, 0, 0, 0, 0, 0)$, $\Phi \neq 0$. From (10) of Lemma 4.3, we have $\Phi^2 = 0$. Choose $P_1 \in (\mathfrak{P}^C)_d$ such that $\Phi P_1 \neq 0$. Now, for $\Theta = \Theta(0, P_1, 0, 0, 0, 0) \in \Theta((\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})$ (Lemma 4.1.(2)), we have

$$(\exp \Theta)R = \left(\Phi, -\Phi P_1, 0, 0, \frac{1}{8}\{\Phi P_1, P_1\}, 0 \right).$$

Hence this case is also reduced to Case (2).

Thus the proof of this proposition is completed. \square

Theorem 4.5. $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} / ((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-} \simeq (\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$.
In particular, $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is connected.

Proof. The group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ acts on the space $(\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$ transitively (Proposition 4.4), so the former half of this theorem is proved. The latter half can be shown as follows. Since $((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})_{1-}$ and $(\mathfrak{W}^C)_{\sigma, \sigma', \mathfrak{so}(8, C)}$ are connected (Propositions 4.2, 4.4), we have that $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is also connected. \square

5. Construction of $Spin(8, C)$ in the group E_8^C

As similar to the group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$, we define the group $(E_8)^{\sigma, \sigma', \mathfrak{so}(8)}$ by

$$(E_8)^{\sigma, \sigma', \mathfrak{so}(8)} = \left\{ \alpha \in E_8 \mid \begin{array}{l} \sigma\alpha = \alpha\sigma, \sigma'\alpha = \alpha\sigma', \\ \Theta(R_{\tilde{D}})\alpha = \alpha\Theta(R_{\tilde{D}}) \text{ for all } \tilde{D} \in \mathfrak{so}(8) \end{array} \right\},$$

where $\mathfrak{so}(8) = (\mathfrak{f}_4)^{\sigma, \sigma'}$.

Proposition 5.1. *The Lie algebra $(\mathfrak{e}_8)^{\sigma, \sigma', \mathfrak{so}(8)}$ of the group $(E_8)^{\sigma, \sigma', \mathfrak{so}(8)}$ is isomorphic to $\mathfrak{so}(8)$ and the Lie algebra $(\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ of the group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is isomorphic to $\mathfrak{so}(8, C)$, that is, we have*

$$(\mathfrak{e}_8)^{\sigma, \sigma', \mathfrak{so}(8)} \cong \mathfrak{so}(8), \quad (\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \cong \mathfrak{so}(8, C).$$

Proof. $(\mathfrak{e}_8)^{\sigma, \sigma', \mathfrak{so}(8)} \cong \mathfrak{so}(8)$ is proved in [4]. The latter case is the complexification of the former case. \square

Theorem 5.2. $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)} \cong Spin(8, C)$.

Proof. The group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is connected (Theorem 4.5) and its type is $\mathfrak{so}(8, C)$ (Proposition 5.1). Hence the group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ is isomorphic to either one of the following groups

$$Spin(8, C), \quad SO(8, C), \quad SO(8, C)/\mathbf{Z}_2.$$

Their centers of groups above are $\mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_2, 1$, respectively. However, we see that the center of $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ has $1, \sigma, \sigma', \sigma\sigma'$, so its center is $\mathbf{Z}_2 \times \mathbf{Z}_2$. Hence the group $(E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ have to be $Spin(8, C)$. \square

6. The structure of the group $(E_8^C)^{\sigma, \sigma'}$

By using the results above, we shall determine the structure of the group $(E_8^C)^{\sigma, \sigma'} = (E_8^C)^\sigma \cap (E_8^C)^{\sigma'}$.

Lemma 6.1. *The Lie algebras $(\mathfrak{e}_8^C)^{\sigma, \sigma'}$ of the groups $(E_8^C)^{\sigma, \sigma'}$ is given by*

$$\begin{aligned} & (\mathfrak{e}_8^C)^{\sigma, \sigma'} \\ &= \{R \in \mathfrak{e}_8^C \mid \sigma R = R, \sigma' R = R\} \\ &= \left\{ (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C \left| \begin{array}{l} \Phi \in (\mathfrak{e}_7^C)^{\sigma, \sigma'}, P = (X, Y, \xi, \eta), \\ Q = (Z, W, \zeta, \omega), X, Y \text{ are diagonal forms,} \\ Z, W \text{ are diagonal forms, } \xi, \eta, \zeta, \omega \in C, \\ r, s, t \in C \end{array} \right. \right\}. \end{aligned}$$

In particular,

$$\dim_C((\mathfrak{e}_8^C)^{\sigma, \sigma'}) = 37 + 8 \times 2 + 3 = 56.$$

Proposition 6.2. *The group $((E_8^C)^{\sigma, \sigma'})_{1_-}$ is a semi-direct product of $\exp(\Theta((\mathfrak{P}^C)_d) \oplus C_-)$ and $(E_7^C)^{\sigma, \sigma'}$:*

$$((E_8^C)^{\sigma, \sigma'})_{1_-} = \exp(\Theta((\mathfrak{P}^C)_d) \oplus C_-) \cdot (E_7^C)^{\sigma, \sigma'}.$$

In particular, $((E_8^C)^{\sigma, \sigma'})_{1_-}$ is connected.

Proof. We can prove this proposition in a similar way to Proposition 4.2. \square

We denote the connected component of $(E_8^C)^{\sigma, \sigma'}$ containing the unit element by $((E_8^C)^{\sigma, \sigma'})_0$.

We define a subspace $(\mathfrak{W}^C)_{\sigma, \sigma'}$ of \mathfrak{W}^C by

$$(\mathfrak{W}^C)_{\sigma, \sigma'} = \{R \in \mathfrak{W}^C \mid \sigma R = R, \sigma' R = R\}.$$

Lemma 6.3. *For $R = (\Phi, P, Q, r, s, t) \in \mathfrak{e}_8^C$ satisfying $\sigma R = R, \sigma' R = R, R \neq 0$, R belongs to $(\mathfrak{W}^C)_{\sigma, \sigma'}$ if and only if R satisfies the following conditions.*

- (1) $2s\Phi - P \times P = 0$ (2) $2t\Phi + Q \times Q = 0$ (3) $2r\Phi + P \times Q = 0$
- (4) $\Phi P - 3rP - 3sQ = 0$ (5) $\Phi Q + 3rQ - 3tP = 0$
- (6) $\{P, Q\} - 16(st + r^2) = 0$
- (7) $2(\Phi P \times Q_1 + 2P \times \Phi Q_1 - rP \times Q_1 - sQ \times Q_1) - \{P, Q_1\}\Phi = 0$
- (8) $2(\Phi Q \times P_1 + 2Q \times \Phi P_1 + rQ \times P_1 - tP \times P_1) - \{Q, P_1\}\Phi = 0$

$$(9) \ 8((P \times Q_1)Q - stQ_1 - r^2Q_1 - \Phi^2Q_1 + 2r\Phi Q_1) + 5\{P, Q_1\}Q - 2\{Q, Q_1\}P = 0$$

$$(10) \ 8((Q \times P_1)P + stP_1 + r^2P_1 + \Phi^2P_1 + 2r\Phi P_1) + 5\{Q, P_1\}Q - 2\{P, Q_1\}Q = 0$$

$$(11) \ 18(\text{ad } \Phi)^2\Phi_1 + Q \times \Phi_1P - P \times \Phi_1Q + B_7(\Phi, \Phi_1)\Phi = 0$$

$$(12) \ 18(\Phi_1\Phi P - 2\Phi\Phi_1P - r\Phi_1P - s\Phi_1Q) + B_7(\Phi, \Phi_1)P = 0$$

$$(13) \ 18(\Phi_1\Phi Q - 2\Phi\Phi_1Q + r\Phi_1Q - t\Phi_1P) + B_7(\Phi, \Phi_1)Q = 0$$

(where B_7 is the Killing form of the Lie algebra \mathfrak{e}_7^C) for all $\Phi_1 \in \mathfrak{e}_7^C, P_1, Q_1 \in \mathfrak{P}^C$.

We denote the connected component of $(E_8^C)^{\sigma, \sigma'}$ containing the unit element by $((E_8^C)^{\sigma, \sigma'})_0$.

Proposition 6.4. *The group $((E_8^C)^{\sigma, \sigma'})_0$ acts on $(\mathfrak{W}^C)_{\sigma, \sigma'}$ transitively. In particular, $(\mathfrak{W}^C)_{\sigma, \sigma'}$ is connected.*

Proof. We can prove this proposition in a similar way to Proposition 4.4 by using Lemma 6.3. \square

Proposition 6.5. $(E_8^C)^{\sigma, \sigma'} / ((E_8^C)^{\sigma, \sigma'})_{1-} \simeq (\mathfrak{W}^C)_{\sigma, \sigma'}$.
In particular, $(E_8^C)^{\sigma, \sigma'}$ is connected.

Proof. The group $(E_8^C)^{\sigma, \sigma'}$ acts on $(\mathfrak{W}^C)_{\sigma, \sigma'}$ transitively (Proposition 6.4), so the former half of this theorem is proved. Since $((E_8^C)^{\sigma, \sigma'})_{1-}$ and $(\mathfrak{W}^C)_{\sigma, \sigma'}$ are connected (Propositions 6.2, 6.4), we have that $(E_8^C)^{\sigma, \sigma'}$ is also connected. \square

Theorem 6.6. $(E_8^C)^{\sigma, \sigma'} \cong (Spin(8, C) \times Spin(8, C)) / (\mathbf{Z}_2 \times \mathbf{Z}_2), \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1), (\sigma, \sigma)\} \times \{(1, 1), (\sigma', \sigma')\}$.

Proof. Let $Spin(8, C) = (E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ (Theorem 5.2) $\subset (E_8^C)^{\sigma, \sigma'}$ and $Spin(8, C) = (F_4^C)^{\sigma, \sigma'}$ (Theorem 2.2) $\subset (E_8^C)^{\sigma, \sigma'}$. We define a mapping $\varphi : Spin(8, C) \times Spin(8, C) \rightarrow (E_8^C)^{\sigma, \sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Since $[R_D, R_8] = 0$ for $R_D \in \mathfrak{spin}(8, C) = \mathfrak{so}(8, C) = (\mathfrak{e}_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ and $R_8 \in \mathfrak{spin}(8, C) = (\mathfrak{f}_4^C)^{\sigma, \sigma'}$, we see that $\alpha\beta = \beta\alpha$. Hence φ is a homomorphism. $\text{Ker } \varphi = \mathbf{Z}_2 \times \mathbf{Z}_2$. In fact, $\dim_C(\mathfrak{spin}(8, C) \oplus \mathfrak{spin}(8, C)) = 28 + 28 = 56 = \dim_C((\mathfrak{e}_8^C)^{\sigma, \sigma'})$ (Lemma 6.1), $\text{Ker } \varphi$ is discrete. Hence $\text{Ker } \varphi$ is contained in the center $z(Spin(8, C) \times Spin(8, C)) = z(Spin(8, C)) \times z(Spin(8, C)) = \{1, \sigma, \sigma', \sigma\sigma'\} \times \{1, \sigma, \sigma', \sigma\sigma'\}$. Among them, φ maps only $\{(1, 1), (\sigma, \sigma), (\sigma', \sigma'), (\sigma\sigma', \sigma\sigma')\}$ to the identity 1. Hence we have $\text{Ker } \varphi = \{(1, 1), (\sigma, \sigma), (\sigma', \sigma'), (\sigma\sigma', \sigma\sigma')\} = \{(1, 1), (\sigma, \sigma)\} \times \{(1, 1), (\sigma', \sigma')\} = \mathbf{Z}_2 \times \mathbf{Z}_2$. Since $(E_8^C)^{\sigma, \sigma'}$ is connected (Proposition 6.5) and again from $\dim_C((\mathfrak{e}_8^C)^{\sigma, \sigma'}) = 56 = \dim_C(\mathfrak{spin}(8, C) \oplus \mathfrak{spin}(8, C))$, we see that φ is onto. Thus we have the required isomorphism $(E_8^C)^{\sigma, \sigma'} \cong (Spin(8, C) \times Spin(8, C)) / (\mathbf{Z}_2 \times \mathbf{Z}_2), \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1), (\sigma, \sigma)\} \times \{(1, 1), (\sigma', \sigma')\}$. \square

7. Main theorem

By using results above, we will determine the structure of the group $(E_8)^{\sigma, \sigma'}$ which is the main theorem.

Theorem 7.1. $(E_8)^{\sigma, \sigma'} \cong (Spin(8) \times Spin(8)) / (\mathbf{Z}_2 \times \mathbf{Z}_2), \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1), (\sigma, \sigma)\} \times \{(1, 1), (\sigma', \sigma')\}$.

Proof. For $\delta \in (E_8)^{\sigma, \sigma'} = ((E_8^C)^{\tau\tilde{\lambda}})^{\sigma, \sigma'} = ((E_8^C)^{\sigma, \sigma'})^{\tau\tilde{\lambda}} \subset (E_8^C)^{\sigma, \sigma'}$, there exist $\alpha \in Spin(8, C) = (E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)}$ and $\beta \in Spin(8, C) = (F_4^C)^{\sigma, \sigma'}$ such that $\delta = \varphi(\alpha, \beta) = \alpha\beta$ (Theorem 6.6). From the condition $\tau\tilde{\lambda}\delta\tilde{\lambda}\tau = \delta$, that is, $\tau\tilde{\lambda}\varphi(\alpha, \beta)\tilde{\lambda}\tau = \varphi(\alpha, \beta)$, we have $\varphi(\tau\tilde{\lambda}\alpha\tilde{\lambda}\tau, \tau\beta\tau) = \varphi(\alpha, \beta)$. Hence

$$\begin{aligned} \text{(i)} \quad & \begin{cases} \tau\tilde{\lambda}\alpha\tilde{\lambda}\tau = \alpha \\ \tau\beta\tau = \beta, \end{cases} & \text{(ii)} \quad & \begin{cases} \tau\tilde{\lambda}\alpha\tilde{\lambda}\tau = \sigma\alpha \\ \tau\beta\tau = \sigma\beta, \end{cases} \\ \text{(iii)} \quad & \begin{cases} \tau\tilde{\lambda}\alpha\tilde{\lambda}\tau = \sigma'\alpha \\ \tau\beta\tau = \sigma'\beta, \end{cases} & \text{(iv)} \quad & \begin{cases} \tau\tilde{\lambda}\alpha\tilde{\lambda}\tau = \sigma\sigma'\alpha \\ \tau\beta\tau = \sigma\sigma'\beta. \end{cases} \end{aligned}$$

Case (i). The group $\{\alpha \in Spin(8, C) \mid \tau\tilde{\lambda}\alpha\tilde{\lambda}\tau = \alpha\} = (Spin(8, C))^{\tau\tilde{\lambda}}$ (which is connected) $= ((E_8^C)^{\sigma, \sigma', \mathfrak{so}(8, C)})^{\tau\tilde{\lambda}}$ (Theorem 5.2) $= (E_8)^{\sigma, \sigma', \mathfrak{so}(8)}$ and its type is $\mathfrak{so}(8)$ (Proposition 5.1), so we see that the group $(E_8)^{\sigma, \sigma', \mathfrak{so}(8)}$ is isomorphic to either one of

$$Spin(8), \quad SO(8), \quad SO(8)/\mathbf{Z}_2.$$

Their centers are $\mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_2, 1$, respectively. However the center of $(E_8)^{\sigma, \sigma', \mathfrak{so}(8)}$ has $1, \sigma, \sigma', \sigma\sigma'$, so the group $(E_8)^{\sigma, \sigma', \mathfrak{so}(8)}$ has to be $Spin(8)$ and its center is $\{1, \sigma, \sigma', \sigma\sigma'\} = \{1, \sigma\} \times \{1, \sigma'\} = \mathbf{Z}_2 \times \mathbf{Z}_2$. From the condition $\tau\beta\tau = \beta$, we have $\beta \in (Spin(8, C))^{\tau} = ((F_4^C)^{\sigma, \sigma'})^{\tau} = (F_4)^{\sigma, \sigma'} = Spin(8)$ ([3] Theorem 1.5). Therefore the group of Case (i) is isomorphic to $(Spin(8) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$.

Case (ii). This case is impossible. In fact, set $\beta = (\beta_1, \beta_2, \beta_3), \beta_k \in SO(8, C)$ satisfying $(\beta_1 x)(\beta_2 y) = \overline{\beta_3(xy)}, x, y \in \mathfrak{C}^C$. From the condition $\tau\beta\tau = \sigma\beta$, we have $(\tau\beta_1, \tau\beta_2, \tau\beta_3) = (\beta_1, -\beta_2, -\beta_3)$. Hence

$$\tau\beta_1 = \beta_1, \quad \tau\beta_2 = -\beta_2, \quad \tau\beta_3 = -\beta_3.$$

From $\tau\beta_1 = \beta_1$, we have $\beta_1 \in SO(8)$, hence β_2 is also $\beta_2 \in SO(8)$ by the principle of triality. Then $-\beta_2 = \tau\beta_2 = \beta_2$ which is a contradiction.

Case (iii) and (iv) are impossible as similar to Case (ii).

Thus we have the required isomorphism $(E_8)^{\sigma, \sigma'} \cong (Spin(8) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2)$, $\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(1, 1), (\sigma, \sigma)\} \times \{(1, 1), (\sigma', \sigma')\}$. \square

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